MODEL ANSWER

AV-8881

B.Sc.-(Hon's) (Fifth semester) Examination,

2015-16

Mathematics: Numerical Analysis

Max: 60

1 (i) $RHS = \frac{1}{2}\delta^2 + \delta\sqrt{\left(1+\frac{\delta^2}{4}\right)}$

$= \frac{(E^{\frac{1}{2}} - E^{-\frac{1}{2}})^2}{2} + (E^{\frac{1}{2}} - E^{-\frac{1}{2}})\sqrt{1+\frac{(E^{\frac{1}{2}}-E^{-\frac{1}{2}})^2}{4}}$

$= \frac{E+E^{-1}}{2} - 1 + \frac{E-E^{-1}}{2} = E - 1 = \Delta$

(ii) Let $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ be the values of $f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$, not necessarily equally spaced, then

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + \cdots$$
$$\cdots + (x-x_0)(x-x_1)\ldots(x-x_{n-1}) f(x_0,x_1,\ldots x_n).$$

(iii) $\Delta^4 u_0 = (E-1)^4 u_0 = (E^4 - 5E^3 + 6E^2 - 4E + 1) u_0 = 29 - 112 + 126 - 44 + 1 = 0.$

(iv) $$y = \frac{y_0+y_1}{2} + \left(u-\frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2}\right]$$
$$+ \frac{u(u-\frac{1}{2})(u-1)}{3!}\Delta^3 y_{-1} + \cdots$$

where $y_0, y_1, \ldots y_n$ be the values of $f(x)$ corresponding to the $x_0, x_1, \ldots x_n$.

(v) $$\int_{x_0}^{x_0+nh} y\,dx = h\left[\frac{y_0+y_n}{2} + (y_1+y_2+\cdots+y_{n-1})\right]$$

where $y=f(x)$ be given for certain equidistant values of x say $x_0, x_0+h, x_0+2h, \ldots$. Let the range (a,b) be divided into n equal parts, each of width h so that $b-a=nh$

(vi) Any two merits and two demerits are considered

(vii) $y_1 = y(x_0+h) = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = h f(x_0, y_0)$, $k_2 = h f(x_0+\frac{1}{2}h, y_0+\frac{k_1}{2})$

$k_3 = h f(x_0+\frac{1}{2}h, y_0+\frac{1}{2}k_2)$

$k_4 = h f(x_0+h, y_0+k_3)$.

(viii) 1, 4, 6 are Eigen values:

Q.2 (a) $RHS = u_x - {}^nC_1 u_{x-1} + {}^nC_2 u_{x-2} - {}^nC_3 u_{x-3} + \cdots$

$$= u_x - {}^nC_1 u_x E^{-1} + {}^nC_2 E^{-2} u_x - {}^nC_3 E^{-3} u_x + \cdots$$

$$= \left(1 - {}^nC_1 E^{-1} + {}^nC_2 E^{-2} - {}^nC_3 E^{-3} + \cdots\right) u_x$$

$$= (1 - E^{-1})^n u_x = \left(1 - \frac{1}{E}\right)^n u_x = \left(\frac{E-1}{E}\right)^n u_x$$

$$= \left(\frac{\Delta}{E}\right)^n u_x = \Delta^n E^{-n} u_x = \Delta^n u_{x-n}$$

(b) Let $f(x)$ be the required function

$$\Delta f(x) = 9x^2 + 11x + 5 \equiv 9x^{(2)} + Ax^{(1)} + B$$

$$= 9x(x-1) + Ax + B$$

Putting $x = 0$, $B = 5$

Putting $x = 1$, $A = 20$

$$\Delta f(x) = 9x^{(2)} + 20x^{(1)} + 5$$

$$f(x) = \frac{9x^{(3)}}{3} + 20 \cdot \frac{x^{(2)}}{2} + 5 \cdot \frac{x^{(1)}}{1} + c$$

where c is a constant. $f(x) = 3x^{(3)} + 10x^{(2)} + 5x^{(1)} + c$

$$= 3x^3 + x^2 + x + c$$

Q. 3(a)

$$y_1 = \frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} + \frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-5-3)} y_{-3}$$

$$+ \frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)} y_3 + \frac{(1+5)(1+3)(1-3)}{(5+5)(5+3)(5-3)} y_5$$

After simplification, we get

$$= y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$$

(b) Here we are given six values, so a polynomial of degree 5 may be fitted which will have its 6th difference is zero

$$\Delta^6 f(x) = 0 \quad \forall x$$

$$\Delta^6 f(x) = (E-1)^6 f(x) = 0$$

$$(E^6 - 6E^5 + 15E^4 - 20E^3 + 15E^2 - 6E + 1) f(x) = 0$$

$$\Rightarrow f(x+6) - 6f(x+5) + 15f(x+4) - 20f(x+3) + 15f(x+2) - 6f(x+1) + f(x) = 0$$

taking $x = 1$, we get $\quad f(5) + f(3) = 152$

taking $x = 2$, we get $\quad 10f(5) + 3f(3) = 1331$

$$f(3) = 27, \quad f(5) = 125.$$

Q.4 (a) Substituting for $\Delta^2 y_0, \Delta^3 y_0, \ldots$ values in Newton's forward interpolation, we get

$$y_p = y_0 + p(\Delta y_0) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \ldots$$

$$= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \ldots$$

Taking Newton's forward formula and substitute $\Delta^2 y_0$, $\Delta^3 y_0, \ldots$ values from difference table, then we get Gauss's forward central difference interpolation formula.

(Complete proof is needed for giving full marks)

(b) Take $x_0 = 11$; $h = 5$; $x_p = 9 \Rightarrow p = \frac{x_p - x_0}{h} = \frac{9-11}{5} = -0.4$

Now the central difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	5225				
		-909			
6	4316		803		
		-1060		-2027	
11	3256		-1224		2961
		-1330		934	
16	1926		-290		
		-1620			
21	306				

From the table, we find

$y_0 = 3256$; $\Delta y_0 = -1330$; $\Delta y_{-1} = -106$

$\Delta^2 y_{-1} = -1224$; $\Delta^3 y_{-1} = 934$; $\Delta^3 y_{-2} = -2027$; $\Delta^4 y_{-2} = 2961$

~~y_p~~ = Stirling ~~form~~ formula (Statement)

After applying values in Stirling formula, we get the required value ~~2961~~ 3725 appr.

Q. 5(a) The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	
1.5	3.375				
		3.625			
2.0	7.0		3		
		6.625		0.75	
2.5	13.625		3.75		0
		10.375		0.75	
3.0	24.0		4.5		0
		14.875		0.75	
3.5	38.875		5.25		
		20.125			
4.0	59				

Here $x_0 = 1.5$ $y_0 = 3.375$ and $h = 0.5$

By Newton's forward difference formula we have

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \dots\right]$$

at $x = 1.5$ $=$ 4.75 (Ans

(b) The Euler-Maclaurin formula is

$$\frac{1}{h}\int_{x_0}^{x_0+nh} f(x)\,dx = \sum_{i=0}^{n} f(x_i) - \frac{1}{2}\left[f x_n + f x_0\right]$$

$$-\frac{h}{12}\left[f'(x_0+nh) - f'(x_0)\right] + \frac{h^3}{720}\left[f'''(x_0+nh) - f'''(x_0)\right] - \dots$$

Putting $fx = x^2$, $f'x = 2x$, $x_0 = 0$, $h = 1$, $x_0 + nh = n$, $x_i = x_0 + ih = i$, we get

$$\int_0^n x^2\,dx = \sum i^2 - \frac{1}{2}\left[n^2 + 0\right] - \frac{1}{12}\left[2n - 0\right]$$

because third and higher order derivatives of x^2 are zero

$$\frac{n^3}{3} = \sum i^2 - \frac{1}{2}i^2 - \frac{n}{6}.$$

$$\sum i^2 = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6} = \frac{n(n+1)(2n+1)}{6}$$

6. General Quadrature formula

Let $I = \int_a^b y\,dx$ where $y = fx$. Let $f(x)$ be given for certain equidistant values of x say $x_0, x_0+h, x_0+2h, \dots$. Let the range (a,b) be divided into n equal parts, each of width h so that $b - a = nh$.

Let $x_0 = a$, $x_1 = x_0 + h = a + h$, $x_2 = a + 2h, \dots, x_n = a + nh = b$.

we ~~as~~ have assumed that the $n+1$ ordinates $y_0, y_1, \dots, y_n$ are at equal intervals

$$I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} y_x\,dx = \int_0 y_{x_0+uh}\, h\,du \quad \text{where } u = \frac{x - x_0}{h},\ dx = h\,du.$$

$$\text{or } I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \dots\right] du$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \dots + (n+1)\text{ terms}\right]$$

taking $n = 3$, then we get

$$\int_{x_0}^{x_0+3h} y\,dx = \cancel{h\left[y_0 + \frac{1}{2}\Delta y_0\right] = h\left[\frac{y_0 + y_1}{2}\right]}$$

$$= h\left[3y_0 + \frac{9}{2}\Delta y_0 + \left(\frac{27}{3} - \frac{9}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{3!}\right]$$

$= h\left[3y_0 + \frac{9}{2}(y_1-y_0) + \frac{9}{4}(y_2-2y_1+y_0) + \frac{3}{8}(y_3-3y_2+3y_1-y_0)\right]$

$= \frac{3h}{8}\left[y_0+3y_1+3y_2+y_3\right]$

IIIly $\displaystyle\int_{x_0+3h}^{x_0+6h} y\,dx = \frac{3h}{8}\left[y_3+3y_4+3y_5+y_6\right]$

. . . .

$\displaystyle\int_{x_0+\overline{n-3}\,h}^{x_0+nh} y\,dx = \frac{3h}{8}\left[y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\right]$

Adding all these integrals where n is a multiple of 3, we have

$$\int_{x_0+0h}^{x_0+nh} y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+\cdots+y_{n-1})+2(y_3+y_6+\cdots+y_{n-3})\right]$$

7. a) Let x_n and x_{n+1} be two successive approximations to the actual root α of $f(x)=0$.

If e_n and e_{n+1} are the corresponding errors.

then we have $x_n-\alpha=e_n$ and $x_{n+1}-\alpha=e_{n+1}$

$$e_{n+1}-e_n = x_{n+1}-x_n = -\frac{f x_n}{f' x_n} = -\frac{f(\alpha+e_n)}{f'(\alpha+e_n)} = -\frac{f(\alpha)+\frac{e_n}{1!}f'(\alpha)+\frac{e_n^2}{2!}f''(\alpha)+\cdots}{f'(\alpha)+\frac{e_n}{1!}f''(\alpha)+\frac{e_n^2}{2!}f'''(\alpha)+\cdots}$$

$$e_{n+1} = e_n - \frac{e_n f'(\alpha)+\frac{e_n^2}{2}f''(\alpha)}{f'(\alpha)+e_n f''(\alpha)} \quad \text{(After simplification)}$$

$= e_{n+1} \propto e_n^2$

The convergence is quadratic and is of order 2.

b) Here $f(x,y) = y+e^x$, $x_0=0$, $y_0=0$ and $h=0.2$

(8) Finding least Eigen value & Eigen vector through any method consider

Using Euler's formula, $y_1^{(0)} = y_0 + h f(x_0,y_0)$

$= 0 + 0.2\, f(0,0) = 0.2$

Now $x_1 = 0.2$ and $f(x_1, y_1^{(0)}) = f(0.2, 0.2)$

$= 0.2 + e^{0.2} = 1.4214$

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(0)})\right] = 0.24214$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(1)})\right] = 0.2463$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(2)})\right] = 0.2468$

$y_1^{(4)} = y_0 + \frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(3)})\right] = 0.2468$